MATH1520 University Mathematics for Applications	Spring 2021

Chapter 6: Application of Derivatives I

Learning Objectives:
(1) Apply L'Hôpital's rule to find limits of indeterminate forms.
(2) Discuss increasing and decreasing functions.
(3) Define critical points and relative/absolute extrema of real functions of 1 variable.
(4) Use the first derivative test to study relative/absolute extrema of functions.

6.1 Limits of indeterminate forms and L'Hôpital's rule

Recall the Remark in the end of Section 2.4 regarding exceptional cases of limits, which can not be computed using the algebraic rules of limits in Proposition 2, but the limits might still exist. Limits of this type are said to be of indeterminate forms.

6.1.1 Limits of indeterminate forms $\frac{0}{0}, \frac{\infty}{\infty}$

Consider $\lim\limits_{x\to a} \frac{f(x)}{g(x)}$,

1. if $\lim\limits_{x\to a} f(x) = A$, $\lim\limits_{x\to b} g(x) = B \neq 0$, $A, B \in \mathbb{R}$, then by the quotient rule,

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim\limits_{x\to a} f(x)}{\lim\limits_{x\to a} g(x)} = \frac{A}{B}.$$

2. if $\lim\limits_{x\to a} f(x) = \lim\limits_{x\to a} g(x) = 0 \quad (\pm\infty)$, then the quotient rule is not applicable. Limits of this type are said to be of indeterminate form type $\frac{0}{0}$ or type $\frac{\infty}{\infty}$

   algebraic rules

   For example,

$$\lim_{x\to 1} \frac{x^2-1}{x^3-1}, \quad \left(\text{type } \frac{0}{0}\right)$$

$$\frac{\lim\limits_{x\to 1}(x^2-1)}{\lim\limits_{x\to 1}(x^3-1)} = \frac{0}{0}$$

$$\lim_{x\to +\infty} \frac{x+1}{2x+3}, \quad \lim_{x\to +\infty} \frac{-x+1}{2x^3}, \quad \left(\text{type } \frac{\infty}{\infty}\right).$$

Theorem 6.1.1 (L'Hôpital's rule for limits of types $\frac{0}{0}, \frac{\infty}{\infty}$). $\frac{\infty}{-\infty}$

Let $f(x)$, $g(x)$ be differentiable and suppose that $g'(x) \neq 0$ near the point a.

If

$$\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0 \quad \text{or} \quad \lim_{x\to a} f(x) = \lim_{x\to a} g(x) = \pm\infty,$$

then

$$\lim_{x\to a} \left(\frac{f(x)}{g(x)}\right) = \lim_{x\to a} \left(\frac{f'(x)}{g'(x)}\right).$$

For limits of indeterminate form $\frac{0}{0}$

$\lim_{x\to a} f(x) = 0 = \lim_{x\to a} g(x)$

Remark. (a) An intuitive explanation: When $f(a) \approx 0 \approx g(a)$,

$\lim_{x\to a} \frac{f}{g}$

$$\underbrace{\frac{f(x)}{g(x)}}_{x\to a} \approx \frac{f(x) - f(a)^{\simeq 0}}{g(x) - \underbrace{g(a)}_{\simeq 0}} = \frac{\frac{f(x)-f(a)}{x-a}}{\frac{g(x)-g(a)}{x-a}}.$$

← lim of this as $x\to a$ is $f'(a)$

(b) The statement of the theorem still holds if "$x \to a$" is replaced by "$x \to \pm\infty$" or "$x \to a^{\pm}$". It also holds if $\lim_{x\to a} f(x) = \pm\infty$ $\lim_{x\to a} g(x) = \mp\infty$. (Use $\lim_{x\to a} \frac{f(x)}{g(x)} = -\lim_{x\to a} \frac{-f(x)}{g(x)}$ and apply the theorem to $\lim_{x\to a} \frac{-f(x)}{g(x)}$.)

$\lim_{x\to\infty} (x^2-1) = +\infty$

$\lim_{x\to\infty} (x^3-1) = +\infty$

$\lim_{x\to\infty} \frac{x^2-1}{x^3-1} = \lim_{x\to\infty} \frac{2}{3x} = 0$

Type $\frac{\infty}{\infty}$, L'Hôpital's rule applies

Example 6.1.1. Limits of type $\frac{0}{0}$

1.

$$\lim_{x\to 1} \left(\frac{x^2-1}{x^3-1}\right) \qquad \text{(check condition 1: } \tfrac{0}{0}\text{)}$$

L'Hôpital's rule

$$= \lim_{x\to 1} \frac{2x}{3x^2} \qquad \text{(check condition 2: this limit is } \tfrac{2}{3}\text{)}$$

$$= \frac{2}{3}.$$

Remark. Alternatively, use the "canceling common factors" trick in the previous chapters.

2.

$$\lim_{x\to 1} \left(\frac{e^x - e}{\sqrt{x} - 1}\right) \qquad \text{(the limit is of type } \tfrac{0}{0}\text{)}$$

$\lim_{x\to 1} (e^x - e) = e - e = 0$

$\lim_{x\to 1} (\sqrt{x} - 1) = 1 - 1 = 0$

L'Hôpital's applies.

$$= \lim_{x\to 1} \frac{e^x}{\frac{1}{2}x^{-\frac{1}{2}}}$$

$$= 2e.$$

algebraic rule works.

$$\frac{\lim_{x\to 1} e^x}{\lim_{x\to 1} \frac{1}{2}x^{-1/2}} = \frac{e}{\frac{1}{2}}$$

3.

$$\lim_{x\to 0^+} \frac{\ln(1+x)}{x^2} \qquad (\text{type } \frac{0}{0})$$
$$= \lim_{x\to 0^+} \frac{\frac{1}{1+x}}{2x}$$
$$= +\infty.$$

$\lim_{x\to 0^+} \ln(1+x) = \ln 1 = 0$

$\lim_{x\to 0^+} x^2 = 0$

L'Hôpital's rule

Example 6.1.2. Limits of type $\frac{\infty}{\infty}$

1.

$$\lim_{x\to+\infty} \frac{-x+1}{2x+3} \qquad (\text{type } \frac{\infty}{\infty})$$
$$= \lim_{x\to+\infty} \frac{-1}{2}$$
$$= -\frac{1}{2}.$$

Remark. The same result can be obtained by dividing both the numerator and the denominator by x.

2.

$$\lim_{x\to+\infty} \frac{\ln x}{x^n}, n \in \mathbb{N} \qquad (\text{type } \frac{\infty}{\infty})$$
$$= \lim_{x\to+\infty} \frac{\frac{1}{x}}{nx^{n-1}}$$
$$= \lim_{x\to+\infty} \frac{1}{nx^n}$$
$$=0.$$

$\lim_{x\to\infty} \ln x = +\infty$

$\lim_{x\to\infty} x^n = +\infty$

L'Hôpital's rule

Remark.

Ex: $\lim_{x\to-\infty} \left(\frac{x^2+1}{x^3}\right) = -\lim_{x\to-\infty} \frac{x^2+1}{-x^3}$ Type "$\frac{\infty}{-\infty}$"

1. L'Hôpital's rule can NOT be applied for determinate form. For example, $\lim_{x\to 1} \frac{x+1}{x+2} = \frac{2}{3}$, but $\lim_{x\to 1} \frac{(x+1)'}{(x+2)'} = \frac{1}{1} = 1$.
2. If $\lim_{x\to a} \frac{f'(x)}{g'(x)}$ is still $\frac{0}{0}$, $\frac{\infty}{\infty}$, then repeat L'Hôpital's rule.
3. L'Hôpital's rule can be used to justify the previous assertion that as $x \to \infty$, higher degree polynomials "grows faster" than lower degree polynomials; exponential functions grow faster than any polynomials; log functions grow slower than any polynomials.

E.g., $\lim_{x\to\infty} \frac{x-1}{x^4+1} = 0$ $x-1$ grows much slower than x^4+1 as $x\to\infty$

E.g., $\lim_{x\to\infty} \frac{x^2}{e^x} = 0$ x^2 grows much slower than e^x as $x\to\infty$

→ E.g. $\lim_{x \to \infty} \frac{\ln x}{x^{\frac{1}{2}}} = 0$ $\ln x$ . . . . . . than $x^{\frac{1}{2}}$ as $x \to \infty$

L'Hôpital

$\lim_{x\to\infty} \frac{\frac{1}{x}}{\frac{1}{2}x^{-1/2}} = 2 \lim_{x\to\infty} x^{-\frac{1}{2}} = 0$

Exercise 6.1.1.

1. $\lim_{x\to 1} \frac{x-1}{\ln x} = 1$
2. $\lim_{x\to +\infty} \frac{x^n}{e^x} = 0$ n is an integer

apply L'Hôpital n times

$\lim_{x\to\infty} \frac{x^2}{e^x} \overset{\text{L'Hôpital}}{=} \lim_{x\to\infty} \frac{2x}{e^x} \overset{\text{L'Hôpital}}{=} 2 \lim_{x\to\infty} \frac{x}{e^x} = 2 \lim_{x\to\infty} \frac{1}{e^x} = 0$

Example 6.1.3. (Applying L'Hôpital's rule twice.)

$$
\begin{aligned}
&\lim_{x\to 0} \frac{e^x - e^{-x} - 2x}{x^2} && (\text{type } \frac{0}{0}) \\
=&\lim_{x\to 0} \frac{e^x + e^{-x} - 2}{2x} && (\text{ still of type } \frac{0}{0}) \\
=&\lim_{x\to 0} \frac{e^x - e^{-x}}{2} \\
=&0
\end{aligned}
$$

6.1.2 Other Indeterminate Forms: $0 \cdot \infty$, $\infty - \infty$, 0^0, 1^∞, ∞^0

All these forms can be converted to forms of types $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

Example 6.1.4. Type $0 \cdot \infty$

$$
\begin{aligned}
&\lim_{x\to 0^+} (x \ln x) && (0 \cdot (-\infty)) \\
=&\lim_{x\to 0^+} \left(\frac{\ln x}{\frac{1}{x}}\right) && \left(\frac{-\infty}{+\infty}\right) \\
=&\lim_{x\to 0^+} \left(\frac{\frac{1}{x}}{-\frac{1}{x^2}}\right) && \text{L'Hôpital} \\
=&\lim_{x\to 0^+} (-x) \\
=&0.
\end{aligned}
$$

$\lim_{x\to 0^+} x = 0$. $\lim_{x\to 0^+} \ln x = (-\infty)$

$\lim_{x\to 0^+} \frac{1}{x} = \infty$

Example 6.1.5. Type $\infty - \infty$

$$\lim_{x\to 0^+}\left(\frac{1}{x}-\frac{1}{e^x-1}\right) \qquad (\infty-\infty)$$
$$=\lim_{x\to 0^+}\frac{e^x-1-x}{x(e^x-1)} \qquad (\frac{0}{0})$$
$$=\lim_{x\to 0^+}\frac{e^x-1}{e^x-1+xe^x} \qquad (\text{ still } \frac{0}{0})$$
$$=\lim_{x\to 0^+}\frac{e^x}{e^x+e^x+xe^x}$$
$$=\frac{1}{2}.$$

$\frac{1}{x}\frac{(e^x-1)}{(e^x-1)} - \frac{x}{x(e^x-1)}$

$\Downarrow$ L'Hospital

$\Downarrow$ L'Hospital

$\lim_{x\to 0^+}(e^x-1-x) = e^0-1-0 = 0$

$\lim_{x\to 0^+}(x(e^x-1)) = 0\cdot(e^0-1)=0$

$\lim_{x\to 0^+}(e^x-1) = 1-1=0$

$\lim_{x\to 0^+}(e^x-1+xe^x) = e^0-1+0 = 0$

Example 6.1.6. Types $1^\infty, \infty^0, 0^0$

Trick: $\boxed{f^g = e^{\ln f^g} = e^{g\ln f}}$

1.

$$\lim_{x\to+\infty} x^{\frac{1}{x}} \qquad (\infty^0)$$
$$=\lim_{x\to+\infty} e^{\ln(x^{\frac{1}{x}})}$$
$$=\lim_{x\to+\infty} e^{\frac{1}{x}\ln x}$$
$$=e^{\lim_{x\to+\infty}\frac{1}{x}\ln x},$$

$x = e^{\ln x}$

because e^y is continuous in y

$$\lim_{x\to+\infty}\frac{1}{x}\ln x \qquad (0\cdot\infty)$$
$$=\lim_{x\to+\infty}\frac{\ln x}{x} \qquad (\frac{\infty}{\infty})$$
$$=\lim_{x\to+\infty}\frac{\frac{1}{x}}{1}$$
$$=0.$$

So,

$$\lim_{x\to+\infty} x^{\frac{1}{x}} = e^0 = 1.$$

2.

$$\lim_{x\to 1^+} x^{\frac{1}{1-x}} \qquad (1^\infty)$$
$$=\lim_{x\to 1^+} e^{\frac{1}{1-x}\ln x}$$
$$=e^{\lim_{x\to 1^+}\frac{\ln x}{1-x}},$$

alg rules don't apply

$\lim_{x\to 1^+} x = 1$ $\qquad \lim_{x\to 1^+}\frac{1}{1-x} = +\infty$

$= \exp\left(\lim_{x\to 1^+}\frac{\ln x}{1-x}\right)$

$$\lim_{x\to 1^+} \frac{\ln x}{1-x} \qquad (\frac{0}{0})$$
$$= \lim_{x\to 1^+} \left(\frac{\frac{1}{x}}{-1}\right) \quad \text{L'Hôpital}$$
$$= -1.$$

$\lim_{x\to 1^+} \ln x = 0$

$\lim_{x\to 1^+} (1-x) = 0$

So,

$$\lim_{x\to 1^+} x^{\frac{1}{1-x}} = e^{-1}. \quad \square$$

3.

$$\lim_{x\to 0^+} x^x \qquad (0^0)$$
$$= \lim_{x\to 0^+} e^{x\ln x}$$
$$= e^{\left(\lim_{x\to 0^+} x\ln x\right)},$$

composition rule for limits doesn't apply

base: $\lim_{x\to 0^+} x = 0$

exponent: $\lim_{x\to 0^+} x = 0$

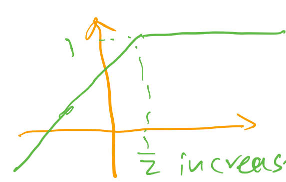
$$\lim_{x\to 0^+} x\ln x \qquad (0\cdot\infty)$$
$$= \lim_{x\to 0^+} \frac{\ln x}{\frac{1}{x}} \qquad (\frac{\infty}{\infty})$$
$$= \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}}$$
$$= \lim_{x\to 0^+} (-x)$$
$$= 0.$$

So,

$$\lim_{x\to 0^+} x^x = e^0 = 1. \quad \square$$

6.2 Monotonicity of Functions and the First Derivative Test

6.2.1 Monotonicity: Increasing/Decreasing Functions



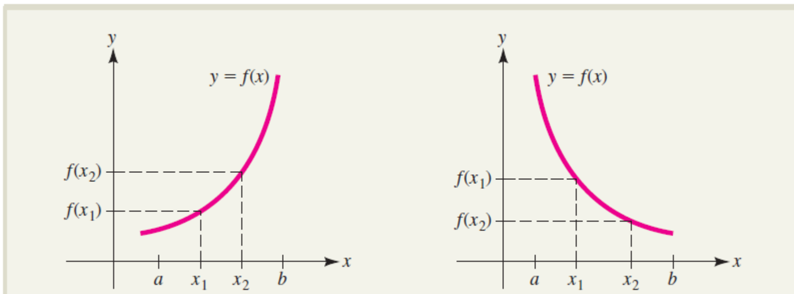
increasing but not strictly increasing

Definition 6.2.1. Let $f(x)$ be a function defined on (a, b). Then

1. $f(x)$ is **increasing** on the interval if $f(x_2) \geq f(x_1)$ whenever $x_2 \geq x_1$.
2. $f(x)$ is **strictly increasing** on the interval if $f(x_2) > f(x_1)$ whenever $x_2 > x_1$.
3. $f(x)$ is **decreasing** on the interval if $f(x_2) \leq f(x_1)$ whenever $x_2 < x_1$.
4. $f(x)$ is **strictly decreasing** on the interval if $f(x_2) < f(x)$ whenever $x_2 > x_1$.

strictly decreasing

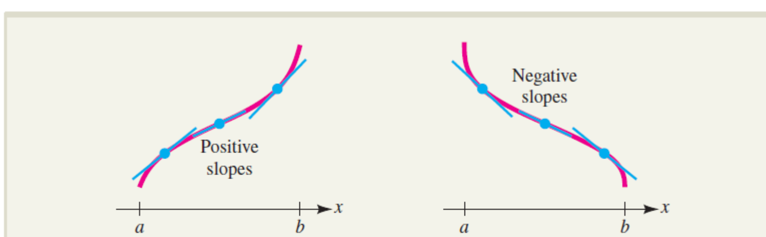
Caveat! The preceding definition is the mathematicians' definition of increasing/decreasing functions. However, some calculus texts define increasing/decreasing functions differently, e.g. [Hoffmann et al.], where "increasing/descreasing functions" refer to the "strictly increasing/descreasing functions" defined above.



(a) $f(x)$ is increasing on $a < x < b$ (b) $f(x)$ is decreasing on $a < x < b$

Theorem 6.2.1. *Let f be a differentiable function on* (a, b).

1. *If $f'(x) \geq 0$ for all $x \in (a, b)$, then $f(x)$ is an increasing function.*
2. *If $f'(x) > 0$ for all $x \in (a, b)$, then $f(x)$ is a strictly increasing function on* (a, b).
3. *If $f'(x) \leq 0$ for all $x \in (a, b)$, then $f(x)$ is a decreasing function.*
4. *If $f'(x) < 0$ for all $x \in (a, b)$, then $f(x)$ is a strictly decreasing function on* (a, b).



(a) $f'(x) > 0$ on $a < x < b$, so $f(x)$ is strictly increasing. (b) $f'(x) < 0$ on $a < x < b$, so $f(x)$ is strictly decreasing.

Example 6.2.1. Show that $f(x) = e^x - x - 1$ is a strictly increasing function on $(0, \infty)$.

Solution. $f'(x) = e^x - 1 > 1 - 1 = 0$. So $f(x)$ is a strictly increasing function. ■

e^0

Remark. Because $f(x)$ is a strictly increasing function, $f(x) > f(0) = 0$ for $x > 0$, i.e.

$$e^x > 1 + x, \text{ for } x > 0.$$

E.g. $f(x) = \ln\left(\frac{1}{x}\right) - x$ $x > 0$

$f' = \frac{1}{\frac{1}{x}}\left(\frac{-1}{x^2}\right) - x = -\frac{1}{x} - 1 < 1$ when $x > 0$.

f' is strictly decreasing on $(0, \infty)$

when $x < -1$
then $f' > 0$. f' is strictly increasing
on $(-\infty, -1)$

Procedure to determine intervals of increase/decrease of f

1. Find all c such that $f'(c) = 0$ or $f'(c)$ is undefined. Divide the line into several intervals.
2. For each intervals (a, b) obtained in the previous step.
   (a) If $f'(x) > 0$, $f(x)$ is a strictly increasing function (↑) on (a, b).
   (b) If $f'(x) < 0$, $f(x)$ is a decreasing function (↓) on (a, b).

Example 6.2.2. Find the intervals in which the function

$$f(x) = 2x^3 + 3x^2 - 12x - 7$$

is strictly increasing/strictly decreasing.

Solution.

$$f'(x) = 6x^2 + 6x - 12 = 6(x+2)(x-1) = 0 \quad \Rightarrow x = -2, 1.$$

So we have 3 intervals: $(-\infty, -2)$, $(-2, 1)$, $(1, \infty)$.

In $(-\infty, -1)$, $x + 1 < 0, x - 1 < 0$, so $f'(x) > 0$.
In $(-1, 1)$, $x + 1 > 0, x - 1 < 0$, so $f'(x) < 0$.
In $(1, +\infty)$, $x + 1 > 0, x - 1 > 0$, so $f'(x) > 0$.

x	$(-\infty, -2)$	-2	$(-2, 1)$	1	$(1, +\infty)$
$f'(x)$	$+$	0	$-$	0	$+$
monotonicity	$\uparrow$		$\downarrow$		$\uparrow$

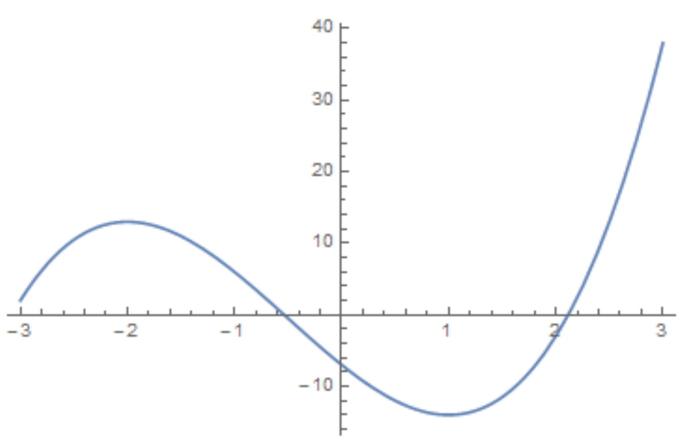


Figure 6.1: $y = 2x^3 + 3x^2 - 12x - 7$